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# Topic 4

## Kirchoff's Laws & Nodal Analysis

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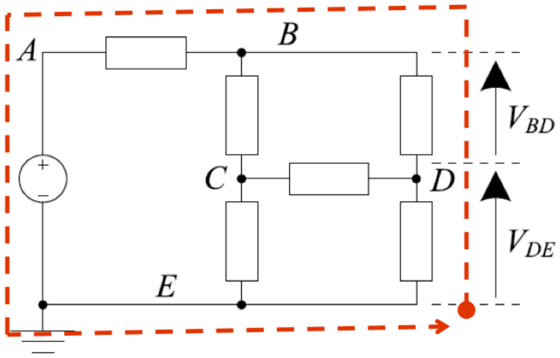
In this lecture, we will consider how to analyse an electrical circuit by applying KVL and KCL.  As  a result, we can predict the voltages and currents around an electrical circuit.  This is a short lecture, but the content is important and fundamental to understanding the basic of electrical engineering  as applied to Design Engineering.

## Kirchoff's Voltage Law

- The five nodes are labelled A, B, C, D, E where E is the reference node.
- Each component that links a pair of nodes is called a *branch* of the network.

**Kirchoff's Voltage Law (KVL)**

The sum of the voltage changes around any closed loop is zero.




- *KVL* is a consequence of the fact that the work done in moving a charge from one node to another does not depend on the route you take; in particular the work done in going from one node back to the same node by any route is zero.
- Example: $V_{DE} + V_{BD} + V_{AB} + V_{EA} = 0$
- Equivalent formulation:

$$V_{XY} = V_{XE} - V_{YE} = V_X - V_Y$$ for any nodes X and Y .

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Consider this circuit here. There is a voltage source providing power to a electrical network consisting of six resistors. This circuit has give nodes labelled A to E. Let us pick node E to be the reference node, so we put a ground symbol on it. You could have picked one of the other nodes as reference. All calculations will give the same results. However, it is often easiest if you pick the –ve node of a voltage source (e.g. a battery) as the reference node) – it usually makes the calculations simpler. Each pair of nodes is connected to the two ends of a component. We call this a branch of the network.

The first law I want you to learn here is the Kirchoff's Voltage Law (or KVL). It says: **If you sum the voltage around ANY closed loop, the sum is always zero.** This is analogous to potential energy in gravitational system we talked about in Lecture 1. Starting from one location on a mountain, you can pick any route up or down. If you return to the place you started from, the potential energy is unchanged. In this case, image you are walking through the network from a node (say A) and go round some branches (component) on the network, and add up all the voltage potential along the way before returning to node A, then the total voltage (or electrical potential energy) will always sum to zero.

Let us consider a loop around the outside of this network as shown in RED. KVL states that summing all the voltages on this loop is zero. The same is true if you take the loop formed by nodes C, D and E. You must be careful with the direction of the arrow. For example, VBD, is voltage at node B relative to node D, the arrow for the voltage is pointing towards B. The current through the branch (i.e. the resistor) is from B to D, i.e. flowing INTO the resistor.

## Kirchoff's Current Law

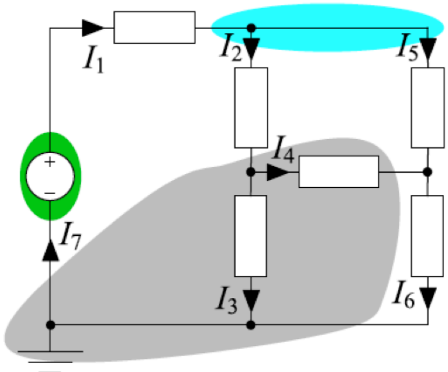
- Wherever charges are free to move around, they will move to ensure charge neutrality everywhere at all times.
- A consequence is ***Kirchoff's Current Law (KCL)*** which says that the current going into any closed region of a circuit must equal the current coming out.

> ***Kirchoff's Current Law (KCL)***
> The currents flowing out of any closed region of a circuit sum to zero.

Green: $I_1 = I_7$

Blue: $-I_1 + I_2 + I_5 = 0$

Gray: $-I_2 + I_4 - I_6 + I_7 = 0$






Remember in Lecture one, we considered how like charges (i.e. sample polarity) repel? The consequence of that is charge generally will NOT accumulate. Therefore if charges are free to move around, neutrally will always be maintained.

This leads to Kirchoff's Current Law (KCL) which is complementary to KVL. It says: **Current going into ANY close region a circuit MUST sum to zero, otherwise charge will accumulate.**

Consider the same network again.

Let us consider a region being a component, say the voltage source (GREEN). Current going in is I7. This must be the same as current coming out, which is I1.

The BLUE region is a circuit node. Current going in is I1 and current coming out is I2 + I5.

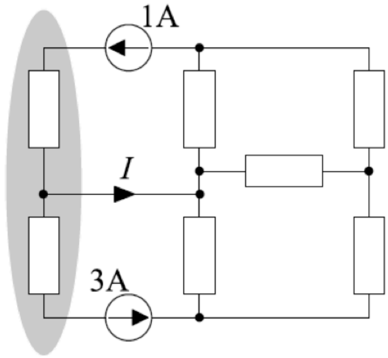
Finally, let us consider the GRAY region, which consists of two registers and three nodes. Current going IN = I2 + I6, and current coming OUT = I4 + I7.

## KCL Example

- The currents and voltages in any **linear circuit** can be determined by using KCL, KVL and Ohm's law.
- Sometimes KCL allows you to determine currents very easily without having to solve any simultaneous equations:
- How do we calculate $I$ ?

  KCL: $-1 + I + 3 = 0$

  $\Rightarrow I = -2\text{A}$




- Note that here I ends up negative which means we chose the wrong arrow direction to label the circuit. This does not matter. You can choose the directions arbitrarily and let the algebra take care of reality.



We have been considering circuits with only resistors, voltage sources and current sources. Such circuits are called "Linear Circuits" because the relationship between currents and voltages in such circuits obey a proportional relationship. We will consider the formal definition of linearity in a later lecture. For now, remember that all currents and voltages in such linear circuits can be determined using KVL, KCL and Ohm's law.

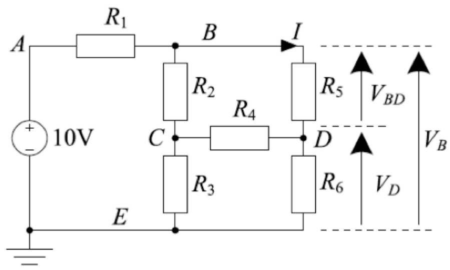
Let us consider the circuit shown here. The unknown current I can be derived easily using KCL because the current IN and OUT of the gray region sums to zero. Therefore I must be -2A. It is negative because in fact the current is flowing INTO the region, which is the opposite direction as indicated by the arrow.

## Aim of Nodal Analysis

- The aim of nodal analysis is to determine the voltage at each node relative to the **reference node** (or ground). Once you have done this you can easily work out anything else you need.
- There are two ways to do this:
  (1) Nodal Analysis - systematic; always works
  (2) Circuit Manipulation - ad hoc; but can be less work and clearer

Reminders:

- A node is all the points in a circuit that are directly interconnected.
- We assume the interconnections have zero resistance so all points within a node have the **same voltage**. Five nodes: $A$, ......... ,$E$.




Ohm's Law: $V_{BD} = I\,R_5$

KVL: $V_{BD} = V_B - V_D$

KCL: Total current exiting (or entering) any closed region is zero.

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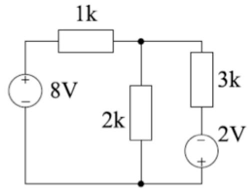
In this lecture, you will learn about a technical known as "**nodal analysis**". If you want to calculate (i.e. predict) voltages and currents in different part of an electrical circuit, you could manipulate the circuits through simplification via substituting components with their equivalent etc. However such approach does not always work.

Nodal analysis is a **systematic way** of analysing a circuit using KCL or KVL, and it always works.

You need to remember what are nodes, KCL, KVL, Ohm's Law and that all interconnections (nodes) have zero resistance.

# Nodal Analysis Stage 1: Label Nodes

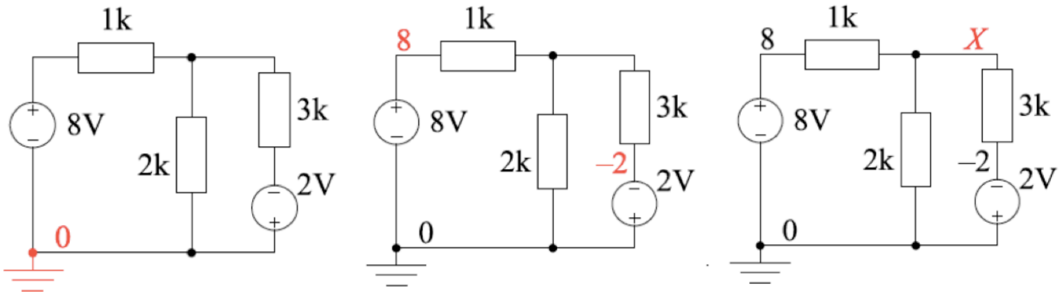
- To find the voltage at each node, the first step is to label each node with its voltage as follows:




(1) Pick any node as the voltage reference. Label its voltage as $0V$.

(2) If any fixed voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end.

(3) Pick an unlabelled node and label it with $X, Y, \ldots$, then go back to step (2) until all nodes are labelled.





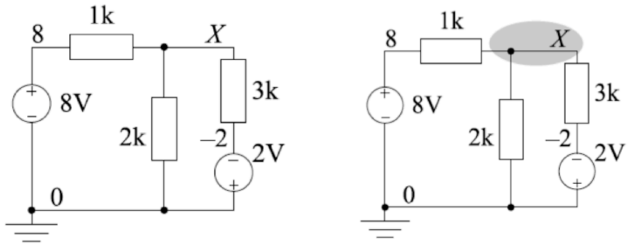

Let us consider a simple circuit as shown here. We need to find voltages at all nodes.

First we pick a node to be reference (or Ground). This is our 0v.

Identify all fixed voltage sources and label these with the voltage values. What remains is one node unlabelled, which we label as X.

## Nodal Analysis Stage 2: KCL Equations

- The second step is to write down a KCL equation for each node labelled with a variable by setting the total current flowing out of the node to zero.
- For a circuit with $N$ nodes and $S$ voltage sources you will have $N - S - 1$ simultaneous equations to solve.




- We only have one variable:

$$\frac{X-8}{1\,\mathrm{k}} + \frac{X-0}{2\,\mathrm{k}} + \frac{X-(-2)}{3\,\mathrm{k}} = 0 \quad \Rightarrow \quad (6X - 48) + 3X + (2X + 4) = 0$$

$$11X = 44 \quad \Rightarrow \quad X = 4$$

Numerator for a resistor is always of the form $X - V_N$ where $V_N$ is the voltage on the other side of the resistor.

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Next we apply KCL to **each** node – i.e. all current at a node sum to zero. A circuit with N nodes and S sources, there will be N-S-1 different unlabelled nodes (S source nodes labelled with fixed voltages, and the 1 is for the 0v reference).

For this circuit we only have one unknown X. Sum all current flowing OUT of X. And we get X (i.e. voltage at node X relative to 0 node) = 4V.

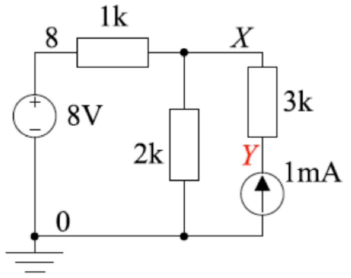
## Current Sources

- Current sources cause no problems.

  (1) Pick reference node.

  (2) Label nodes: 8, X and Y .

  (3) Write equations

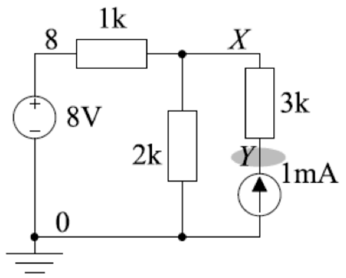



$$\frac{X-8}{1} + \frac{X}{2} + \frac{X-Y}{3} = 0$$

$$\frac{Y-X}{3} + (-1) = 0$$

Ohm's law works OK if all resistors are in kΩ and all currents in mA.

(4) Solve the equations: X = 6, Y = 9






Let us replace one of the voltage source with a current source. We can use the same method to analyse this circuit.

Pick the reference node and label this.

Label the other nodes 8, X and Y.

Now use KCL at X and Y, and we get the equations shown.

Since all resistors are in kilo ohms, and all currents are in mA, with Ohm's Law, V=IR, therefore k x m = 1. We can ignore the multipliers – it will work.

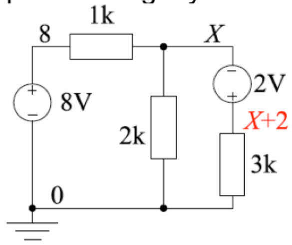
Solving the simultaneous equation gives us X = 6 and Y = 9 (both in V).

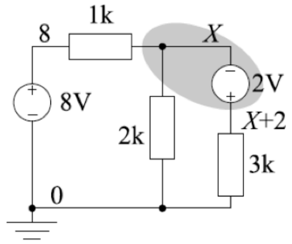
## Floating Voltage Sources

- ***Floating voltage sources*** have neither end connected to a known fixed voltage. We have to change how we form the KCL equations slightly.
  - (1) Pick reference node.
  - (2) Label nodes: 8, $X$ and $X+2$ since it is joined to X via a voltage source.
  - (3) Write KCL equations but count all the nodes connected via floating voltage sources as a single "*super-node*" giving one equation

$$\frac{X-8}{1}+\frac{X}{2}+\frac{(X+2)-0}{3}=0$$

  - (4) Solve the equations: X = 4




Ohm's law always involves the difference between the voltages at either end of a resistor. (Obvious but easily forgotten)

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So far the voltage sources have been connected to 0V reference at one end. That makes the calculation easy because both end of the source have known voltages.

If the voltage source do not have either end connected to a known voltage, it is called a floating voltage source.

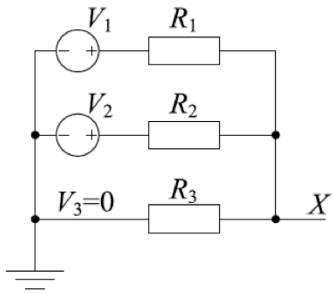
Handling such a source is easy. Simply label one end as an unknown voltage, say X, and the other end is related to X. In the example shown here, the negative end of the source is labelled X. The positive end is simply X+2.

When apply KCL to analyse the circuit, instead of summing current at a node, apply KCL to the entire source shown in gray. This region is treated as a super node. Applying this method results in having only one equation for this circuit.

## Weighted Average Circuit

- A very useful sub-circuit that calculates the weighted average of any number of voltages.
- KCL equation for node $X$:

$$\frac{X-V_1}{R_1} + \frac{X-V_2}{R_2} + \frac{X-V_3}{R_3} = 0$$




- Still works if $V_3 = 0$.
- Or using conductances:

$$(X - V_1)G_1 + (X - V_2)G_2 + (X - V_3)G_3 = 0$$

$$X(G_1 + G_2 + G_3) = V_1G_1 + V_2G_2 + V_3G_3$$

$$X = \frac{V_1G_1+V_2G_2+V_3G_3}{G_1+G_2+G_3} = \frac{\sum V_iG_i}{\sum G_i}$$

Voltage X is the average of V1, V2, V3 weighted by the conductances.



Consider another example circuit. This one is very useful because it produces a WEIGHTED AVERAGE of a number of voltage sources.

The algebraic manipulation produces an output X, which is the average of V1, V2 and V3 weighted by the conductance of each branch.

$X=V_1 G_1 /G_1 +G_2 +G_3 +V_2 G_2 /G_1 +G_2 +G_3 +V_3 G_4 / G_1 +G_2 +G_3$

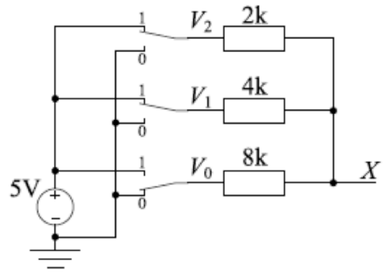
## Simple Digital to Analogue Converter

A 3-bit binary number, $b$, has bit-weights of $4$, $2$ and $1$. Thus $110$ has a value $6$ in decimal. If we label the bits $b_2 b_1 b_0$, then $b = 4b_2 + 2b_1 + b_0$.

We use $b_2 b_1 b_0$ to control the switches which determine whether $V_i = 5\,\mathrm{V}$ or $V_i = 0\,\mathrm{V}$. Thus $V_i = 5b_i$. Switches shown for $b = 6$.

$$X = \frac{\frac{1}{2}V_2 + \frac{1}{4}V_1 + \frac{1}{8}V_0}{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$$

$$= \frac{1}{7}\left(4V_2 + 2V_1 + V_0\right)$$




but $V_i = 5 \times b_i$ since it connects to either $0$ V or $5$ V

$$= \frac{5}{7}\left(4b_2 + 2b_1 + b_0\right) = \frac{5}{7}b$$

$$G_2 = \frac{1}{R_2} = \frac{1}{2\,k} = \frac{1}{2}\,\mathrm{mS}, \ \ldots$$

So we have made a circuit in which $X$ is proportional to a binary number $b$.



This simple circuit can be used to produce a simple digital-to-analogue converter (DAC). Consider the circuit shown above. The three digital switches are controlled by three binary bits: $b_2\ b_1\ b_0$. When the digital bit is high $b_i$ ('1'), the switch $SW_i$ is connected to 5v voltage source, otherwise it is connect to ground.

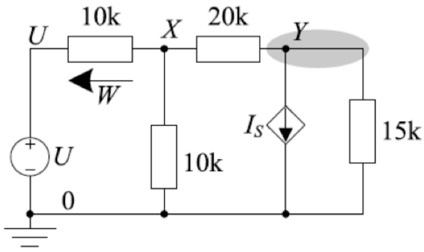
Using the results from last slide, we can see that the voltage X is given by:

$$X = \frac{\frac{1}{2}V_2 + \frac{1}{4}V_1 + \frac{1}{8}V_0}{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}} = \frac{1}{7}\left(4V_2 + 2V_1 + V_0\right)$$

Therefore X is proportional to the digital value of the binary number $b_2\ b_1\ b_0$.

## Dependent Voltage Sources

- A *dependent* voltage or current source is one whose value is determined by voltages or currents elsewhere in the circuit. These are most commonly used when modelling the behaviour of transistors or op-amps. Each dependent source has a *defining equation*.
- In this circuit: $I_S = 0.2W$ mA where $W$ is in volts.




(1) Pick reference node.

(2) Label nodes: $0, U, X$ and $Y$.

(3) Write equation for the dependent source, $I_S$, in terms of node voltages:

$$I_S = 0.2\ (U - X)$$

(4) Write KCL equations:

$$\frac{X-U}{10} + \frac{X}{10} + \frac{X-Y}{20} = 0 \qquad \frac{Y-X}{20} + I_S + \frac{Y}{15} = 0$$

(5) Solve all three equations to find $X$, $Y$ and $IS$ in terms of $U$:

$$X = 0.1U,\ Y = -1.5U,\ I_S = 0.18U$$

Note that the value of $U$ is assumed to be known.

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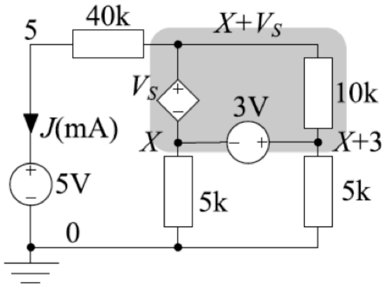
Here is a new component known as **DEPENDENT source** (voltage or current). Instead of having a fixed voltage or current value, its output value is determined by voltage or current **elsewhere in a circuit**. This is often used to model the behaviour of more complicated circuits such as an amplifier or even an entire system. We will come back to modeling and amplification in a later lecture.

Each dependent source has a defining equation which provide the relationship between the source value and the causal parameter. In the circuit shown, the current source IS is controlled by the voltage W across the 10k resistor.

Applying KCL at X and Y. We have three equations (the defining equation for the current source and two KCL equations) and four unknowns: U, IS, X and Y. Solving the three equations provides a solution in terms of U, the voltage source value, which is assumed to be known.

## Dependent Voltage Sources

- The value of the highlighted dependent voltage source is $V_S = 10J$ Volts where $J$ is the indicated current in mA.




(1) Pick reference node.

(2) Label nodes: $0, 5, X, X+3$ and $X+V_S$ .

(3) Write equation for the dependent source, $V_S$, in terms of node voltages:

$$V_S = 10J = 10 \times \frac{X+V_S-5}{40} \Rightarrow 3V_S = X-5$$

(4) Write KCL equations: all nodes connected by floating voltage sources and all components connecting these nodes are in the same "super-node"

$$\frac{X+V_S-5}{40} + \frac{X}{5} + \frac{X+3}{5} = 0$$

(5) Solve the two equations: $X = -1$ and $V_S = -2$



Here is a more complex example with dependent voltage sources. The dependent voltage source $V_S$ is governed by the current J. Remember, we generally use units of mA for currents, k ohms for resistors and V for voltages.

First we write an equation for the dependent voltage source in terms of node voltage in step 3).

Next we create a "**super-node**" that includes the floating voltage source AND all components connected to the source $V_S$. The two nodes of the source are X and X+$V_S$. Here we include the 3V fixed source and the 10k resistor because including these components do not add any new unknown variables.

Now apply KCL to this super-node (or region) and produces one equation in terms of X and $V_S$ only. Two equations, two unknowns, we have our solution.

# Universal Nodal Analysis Algorithm

① Pick any node as the voltage reference. Label its voltage as 0 V. Label any dependent sources with VS, IS, . . ..

② If any voltage sources are connected to a labelled node, label their other ends by adding the value of the source onto the voltage of the labelled end.

③ Pick an unlabelled node and label it with X, Y, . . ., then loop back to step (2) until all nodes are labelled.

④ For each dependent source, write down an equation that expresses its value in terms of other node voltages.

⑤ Write down a KCL equation for each "normal" node (i.e. one that is not connected to a floating voltage source).

⑥ Write down a KCL equation for each "super-node". A super-node consists of a set of nodes that are joined by floating voltage sources and includes any other components joining these nodes.

⑦ Solve the set of simultaneous equations that you have written down.



Let me recap. Here are the steps in nodal analysis. You would have realized that we have only used KCL in this example. You could have used KVL (summing voltages in a closed loop equals zero) and have the same results. However, generally you will find that using KCL and summing current is slightly easier to compute.

Doing nodal analysis is actually easier than it first appears. However it requires practice. I have provided a large number of example questions in the tutorial problem sheet 2 for you to try. The trick is to "*see*" the circuit in a way that makes the circuit as simple as possible. Then work out how many unknowns there are, and which are the simplest equations to produce in order to solve the unknowns.

## Summary

- Nodal Analysis
  - Simple Circuits (no floating or dependent voltage sources)
  - Floating Voltage Sources
    - use supernodes: all the nodes connected by floating voltage sources (independent or dependent)
  - Dependent Voltage and Current Sources
    - Label each source with a variable
    - Write extra equations expressing the source values in terms of node voltages
    - Write down the KCL equations as before